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Bhāskara II's Derivation for the Surface of a Sphere

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The famous भास्कराचार्य Bhāskarācārya (born A. D. 1114), son of Maheśvara, was a great Indian astronomer and mathematician. He is now usually designated as Bhāskara II to distinguish him from his name-sake, Bhāskara I, who was active in the early part of the seventh century A. D. The Līlāvatī (लीलावती)o f Bhāskara II is the most popular book on ancient Indian mathematics and is devoted to the elementary mathematics (Arithmetic, Mensuration, and etc.)[1]. It was translated into Persian by Faizi in 1587.

Bhā kara II also wrote an important treatise on "Algebra" (बीजगणित) along with his own commentary on it, His voluminous astronomical work सिद्धान्तशिरोमणि Siddhānta--śiromaṇi (=SS) was composed in A. D. 1150 and was commented by the author himself[2]. This commentary is usually called the वासनाभाष्य Vāsana-Bhāṣya (=VB). The composition of some other works is also attributed to him[3].

Two centuries earlier than Bhāskara II, there lived another Indian astronomer called Āryabhata II (A.D. 950). In his Mahā-Siddhānta, XVI, 38, Āryabhaṭa II gives the following rule[4]

पर्ि धिघ्नो व्यासः स्यात् कन्दुकजालोपमं कुपृष्ठफलम् ॥ ३८ ॥

Paridhighno vyāsah syāt kandukajālopamaṁ kupṛṣthaphalam //38//
'(Earth's) circumference multiplied by (its) diameter becomes the Earth's surface-area like the (area of the) net covering a ball.' That is,

surface of a sphere = circumference × diameter or $S=C\times D=4\pi R^2$... (1)

where S, C, D, R are the surface area, circumference, diameter and radius respectively.

An equivalent of the rule (1) has been given later on by Bhāskara II in his Līlāvatī[5]. In the third chapter, called Bhuvana-Kośa, of the Golādhyāya part of his SS and the VB there on, the author discusses the topic in more details. SS, Gola., III, 52 contains a statement of the rule (1).

The VB (p. 187) under SS, Gola., III, 54-57 quotes the following incorrect rule from Lalla (eighth century)

वृत्तफलं परिधिघ्नं समंततो भवति गोलपृष्ठफलम्

Vṛttaphalaṁ paridhighnaṁ samaṁtato bhavati golapṛsthaphalam.
This text has been interpreted to mean that:

'The area of the circle (greatest section of a sphere) multiplied by the circumference becomes, the area of the surface of sphere'.

That is,

$$S=\pi R^2\times 2\pi R=2\pi^2R^3.$$

This formula is obviously very wrong and so it has been vehemently criticised by Bhāskara II.

If Lalla, who knew the work of Āryabhata I (born A. D. 476), was not aware of the correct rule for the surface of a sphere, we may assume that Āryabhata I also did not know the same. However, some attempt has been made to credit Āryabhata I with the knowledge of the formula (1) by giving a peculiar interpretation to a rule found in his Āryahhatīya II (Gaṇita), verse no. 7, second half[6].

Bhāskara's VB (pp. 187-188) under SS, Gola, III, 54-57 also contains a derivation of the rule (1) by using a sort of crude integration. A some what free translation of the relevant Sanskrit text may be given as follows:

Make a model of the Earth in clay or wood and suppose its circumference to be equal to the minutes in a circle, that is, 21600 units. Mark a point on the top of it. With that point as the centre and with the (arcual) radius equal to the ninety-sixth part of the circumference, that is, 225 minutes ($=h$ say); describe a circle. Again with the same centre, with twice that (arcual) radius describe another circle, with three times that, another circle; and so on till 24 times. Thus there will be 24 (horizontal) circles.

The radii of these circles will be the (corresponding 24 tabular) Sines 225 etc. (that is, R sin h which is equal to 225 to the nearest minute, R sin $2h$, R sin $3h$,...upto R sin $24h$ which is eqal to R itself). From them the lengths of the circles can be determined by proportion. There the length of the last circle is equal to the minutes in a circle, that is, 21600, and its radius is equal to the Trijyā (Sine of three signs or Sine of 90 degrees, that is Sinus totus), that is, 3438. The above Sines (or radii) multiplied by the minutes in a circle and divided by the Sinus totus become the lengths of the (corresponding) circles.

Between any two (consecutive) circles there is an annular figure in the form of a belt. They are 24 in number. There will be more when more tabular Sines are used (that is, when finer interval is taken).

In each annulus (imgined to be a trapezium), the larger lower circle may be supposed to be the base, the upper smaller circle as the face (or top) and 225 (that is, the common arcual distance h) as the altitude. Thus by the rule "altitude multiplied by half the sum of the base and the face (that is, the rule for the area of a trapezium)" we get the areas of the annular figures separately. The sum of those areas is the surface area of half the sphere, Twice that is the surface area of the whole sphere. That indeed is equal to the product of the diameter and the circumference.'

Let the circumferences of the circles starting from the top be $C_1, C_2, \ldots C_{24}$ and the areas of the corresponding belts (with above circles as their respective lower edges) be $A_1, A_2, \ldots A_{24}$.

We have

$$A_1 = (h/2)\ (0 + C_1)$$

$$A_2 = (h/2)\ (C_1 + C_2)$$

$$A_3 = (h/2)\ (C_2 + C_3)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$A_{24} = (h/2)\ (C_{23} + C_{24})$$

Therefore, the surface area of the whole sphere will be given by

$$S = 2\ (A_1 + A_2 + \ldots + A_{24})$$

$$= 2h\ (C_1 + C_2 + \ldots + C_{23} + \tfrac{1}{2}\ C_{24})$$

$$= 2h \times 21600\ (S_1 + S_2 + \ldots + S_{24} - \tfrac{1}{2}\ R)/R,$$

where $S_1, S_2, \ldots$ are the tabular Sines.

Now Bhāskara himself gives (VB, p. 189) the value of the bracketed quantity needed above to be 52514. Using this we get

$$S = 21600 \times 2 \times 225 \times 52514/3438$$

$$= 21600 \times 2 \times 3437 \text{ nearly}$$

$$= \text{circumference} \times \text{diameter, practically.}$$

In connection with this derivation, Sengupta[8] has remarked that "although we miss here the highly ingeneous method of Archimedes (born 287 B. C.) in summing up a trigonometrical series, there can be no question that the Indian method is perfectly original".

Before concluding it may be mentioned that Bhāskara II has also given an alternate procedure to derive the formula for the surface of a sphere by dividing the surface into lunes (vaprakas) like the natural divisions of the fruit of myrobalan (आँवला) in his SS Gola., III, 58—61 and the VB (pp. 188—89) there upon.

References and Notes

1. H. T. Colebrooke's English transl. (1817) of the work has been again reprinted by M/s Kitab Mahal, Allahabad, 1967.

2. The astronomical work is in two parts namely, Graha-ganita and Golādhyāya. Here

we are using Bapu Deva Sastri's edition of the work along with the commentary, Kashi Sanskrit Series No, 72, Benares, 1929.

3. Bhāskara II wrote a manual of astronomy called करणकुतूहल Karāna Katūhala, or ब्रह्मतुल्य Brahmatulya (A. D. 1183 ?); a Commentary on Lalla's astronomical work (see K. S. Shukla's edition of Pātiganita of Śrīdharācārya Lucknow University, Lucknow, 1959, p. XXII). His other possible minor works may be सर्वतोभद्रयन्त्र and वसिष्ठतुल्य (see S. Dvivedi's Ganaka-Tarangini, Benares, 1933, p. 35). His authorship of the बीजोपनय was doubted by S. R. Das (see H. R. Kapādia's edition of the Ganita-Tilaka, Oriental Institute, Baroda, 1937, p. L XIII) and has been refuted by T. S. Kuppanna Sastri ("The Bijopanaya ; Is it a work of Bhāskārācārya", J. Oriental Institute Vol. 8, 1959, pp. 399-409).

4. S. Dvivedi's edition, Braj Bhusan Das & Co., Benares, 1910, fasciculus II, p. 192.

5. See Colebrooke's transl., Op. cit., Rule 203, p. 117.

6. See Kurt Elfering's German article in Rechenpfennige (Felicitation Volume presented to Dr. Vogel), Deutschen Museum, Munich 1968, pp. 57—67.

7. The value 52514 given by Bhaskara II is on the basis of the Sine tables found in the works like Āryabhatīya, Sūrya-Siddhānta and Lalla's Śiṣyadhivṛddhida. Otherwise, on the basis of the Sine table found in the Mahā-Siddhānta or that which is given by Bhāskara II himself, the value should be 52513. However, the difference is insignificant here.

8. P. C. Sengupta : "Infinitesimal Calculus in India-Its Origin and Development". J. Dept. of letters (Calcutta University), Vol. XXII (1932), article no. 5, p. 17.