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The Hindu Method of Solving Quadratic Equations

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'Is it that the Greeks had such a marked contempt for applied Science, leaving even the instruction of their children to slaves? But if so, how is it that the nation that gave us geometry and carried this science so far did not create even rudimentary algebra? Is it not equally strange that algebra, that corner-stone of modern mathematics, also originated in India, and at about the same time that positional numeration did?'

(Quoted by Jawahar Lal Nehru in his "Discovery of India" p. 211)

Actually the modern algebra can be said to begin with the proof by Paolo Ruffini (1799) that the equation of fifth degree can not be solved in terms of the radicals of the coefficients*. Before this time algebra was mostly identical with the solution of equations. The method of solving a quadratic equation by completing the square is called the Hindu Method. It is based on Sridhara's Rule. The book of Sridhara (circa 750 A. D.) in which this rule was extant not extinct. But many subsequent authors quoted the rule and attributed it to Sridhara. One such author is Gyanraj (cira 1503 A.D.) who gives the rule, in his algebra, as

चतुराहत वर्ग समै रूपैः पक्षद्वयं गुणयेत् ।
अव्यक्त वर्ग रूपैर्युक्तौ पक्षौ ततो मूलम् ॥

"Multiply both the sides by a quantity equal to four times the coefficient of the square of the unknown; add to both sides a quantity equal to the square of the coefficient of the unknown; then take the root".

Let us apply the rule to solve the following quadratic equation (written in the ancient way)

$$ax^2+bx=c \quad \ldots \quad \ldots \quad (1)$$

Multiplying both sides by 4a,

$$4\,a^2x^2+4\,abx=4\,ac$$

Adding b^2 to both sides,

$$4\,a^2x^2+4\,abx+b^2=b^2+4\,ac$$

i.e. $(2\,ax+b)^2=b^2+4ac \quad \ldots \quad (2)$

Taking root on each side

$$2\,ax+b=\sqrt{b^2+4\,ac}$$

$$\text{Hence } x=\frac{\sqrt{b^2+4\,ac}-b}{2\,a} \quad \ldots \quad (3)$$

The solution (3) can be written down by employing the Rule of Brahmagupta (c. 628 A.D.) given in his Brahma Sphuta Siddhanta (XVIII—44) and is equivalent to:

"The quadratic: The absolute quantity (i.e. c) multiplied by four times the coefficient (i.e.a) of the square of the unknown, is increased by the square of the coefficient (i.e. b) of the unknown; the square root of the result being diminished by the coefficient of the unknown, is the root."

Now the quantity $(2\,ax+b)$ is the differential coeff. of the L.H.S. of (1) and the quantity $(b^2+4\,ac)$ is called the Discriminant of the equation (1). The intermediary step (2) can be written down immediately by applying the aphorism

चलित कलित वर्गौ विवेचकः

'Differential-coefficient-square (equals) discriminant'.

*After this demonstration of the insolvability of quintic equation algebraically, a transcendental solution involving elliptic integrals was given by Hermite in Comptes Rendus (1858).

This aphorism was disclosed by Jagadguru swami Shri Bharti Krishna Tirtha, Sankaracharya of Govardhan Math, Puri, during the lectures on his interpretation of the Vedas in relation to Mathematics at the Banaras Hindu University in 1949.

Nature and Number of roots :

'The Hindus early saw in "opposition of direction" on a line an interpretation of positive and negative numbers. In Europe full possession of these ideas was not acquired before Girard and Descartes (17th century)' (Cajori p. 233). Hindus were the first to recognise the existence of absolute negative numbers and of irrational numbers. (Cajori p. 101) A Babylonian of sufficiently remote time, who gave 4 as the root of

$x^2 = x + 12$

had solved his equation completely because negative numbers were not in his number-system (Bell p. 11). Diophantus of Alexandria the famous Greek algebraist regarded

$4x + 20 = 4$

as "absurd" (Kramer p. 99) And although he solved equations of higher degrees but accepted only positive roots. Because of his lack of clear conception of negative numbers and algebraic symbolism he had to study the quadratic equation separately under the the three types :

(i) $ax^2 + bx = c$

(ii) $ax^2 = bx + c$

(iii) $ax^2 + c = bx$

taking the coefficients always positive.

'Cardan (1501-1576) called negative roots of an equation as "fictitious" and positive roots as "real" (Cajori p. 227). Vieta (1540-1603) is also stated to have rejected all, except positive roots of an equation. (Ibid p. 230). In fact, as Cajori writes, 'Before 17th century the majority of the great European algebraists had 'not quite risen to the views taught by Hindus'. (Ibid p. 233). Hindus recognised two answers for quadratic equation. Thus Bhaskara gave 50 and—5 as roots of

$x^2 - 45x = 250$

'And the most important advance is the theory of quadratic equations made in India is unifying under one rule the three cases of Diophantus' (Ibid p. 102)

'The first clear recognition of imaginaries was Mahavira's extremely intelligent remark (9th century) that, in the nature of things, a negative number has no square root. Cauchy made same observation in 1847' (Bell p. 175). For Mahavira cf :—

धनं धनर्णयो वर्गौ मूलं स्वर्णे तयोः क्रमात् ।
ऋणं स्वरुपतोऽ वर्गो यतस्तस्मान्न तत्पदम् ॥

(Mahavira's Ganita-sara sangraha Chapter I, Verse No.52)

Same ideas are found in other Hindu Mathematics books. We again quote Cajori—'An advance far beyond the Greeks is the statement of Bhaskara that "the square of a positive number, as also of a negative number, is positive; that the square root of a positive number is two-fold positive and negative. There is no square root of a negative number, for it is not a square" (p. 102).

The Greeks sharply discriminated between numbers and magnitudes, that the irrational was not recognised by them as a number. By Hindus irrationals were subjected to the same process as ordinary numbers and were indeed regarded by them as numbers. By doing so they greatly aided the progress of Mathematics.

From the foregoing statements of the facts of History of Mathematics the reader will easily agree with Hankel with whose quotation I close this small article :

"If one understands by algebra the application of arithmetical operations to complex magnitudes of all sorts, whether rational or irrational numbers or space-magnitudes, then the learned Brahmins of Hindustan are the real inventor of Algebra" (Cajori p. 195).

General Bibliography

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* This Revered Swami used to say that he had written 16 volumes, one of each of the sixteen sutras, and that the MSS of these volumes were doposited at the house of one of his disciples. Unfortunately, the said manuscripts were lost (Vedic Math. p. X).